ON THE VANISHING OF THE LOWER K-THEORY OF THE HOLOMORPH OF A FREE GROUP ON TWO GENERATORS

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ABSTRACT. We show that the holomorph of the free group on two generators satisfies the Farrell–Jones Fibered Isomorphism Conjecture. As a consequence, we show that the lower K-theory of the above group vanishes.

1. Introduction

Certain obstructions that appear in problems of topological rigidity of manifolds are elements of algebraic K-groups, specially lower K-groups. For this reason, the calculation of the lower K-groups has implications in geometric topology.

The main modern tool for calculating lower K-groups (and other geometrically interesting obstruction groups) is the Farrell–Jones Fibered Isomorphism Conjecture. The Conjecture provides an inductive method for calculating the obstruction groups of a group from those of certain subgroups. More specifically, if a group satisfies the Fibered Isomorphism Conjecture for a specific theory, then the obstruction groups can be calculated from the obstruction groups of the *virtually cyclic subgroups*. The last class of subgroups consists the finite subgroups and groups that are virtually infinite cyclic. The virtually infinite cyclic groups are of two types:

- Groups V that surject onto the infinite cyclic group $\mathbb{Z}$ with finite kernel, i.e. $V = H \rtimes \mathbb{Z}$ with H finite.
- Groups W that surject onto the infinite dihedral group D_∞ with finite kernel, i.e. $W = A *_B C$ with B finite and $[A : B] = [C : B] = 2$.

Obstruction groups that can be calculated this way are pseudoisotopy groups, K-groups of group rings, K-groups of C^*-algebras, $L^{-\infty}$-groups.

The fundamental work of Farrell–Jones ([8]) deals with the Fibered Isomorphism Conjecture for the pseudoisotopy spectrum. It should be noted that if a group satisfies the Fibered Isomorphism Conjecture for pseudoisotopies, then it satisfies the Isomorphism Conjecture for the lower K-groups. The reason for this is that the lower homotopy groups of the pseudoisotopy spectrum and the K-theory spectrum are isomorphic. For K-groups there is a refinement of the Conjecture that was given in [6]. They showed that finite and virtually infinite cyclic subgroups of the first type suffice in detecting the K-theory of the group.

Our main interest is in computing the lower K-groups of the holomorph of F_2, the free group on two generators. For a group G, the holomorph of G is the universal split extension of G. Thus, it fits into a split exact sequence:

$$1 \to G \to \mathrm{Hol}(G) \to \mathrm{Aut}(G) \to 1.$$

The main result of the paper is the following:

Theorem (Main Theorem)**.** *The group* $\mathrm{Hol}(F_2)$ *satisfies the Fibered Isomorphism Conjecture for pseudoisotopies. Furthermore, if* $\Gamma < \mathrm{Hol}(F_2)$ *then*

$$Wh(\Gamma) = \widetilde{K}_0(\mathbb{Z}\Gamma) = K_i(\mathbb{Z}\Gamma) = 0, \quad \textit{for } \ i \leq -1.$$

Notice that $\mathrm{Hol}(F_2)$ is equipped with a sequence of surjections:

$$\mathrm{Hol}(F_2) \to \mathrm{Aut}(F_2) \to \mathrm{GL}_2(\mathbb{Z}).$$

The second surjection is induced by sending an automorphism of F_2 to an automorphism of its abelianization $\mathbb{Z}^2$. Notice that the kernel of both surjections is isomorphic to F_2. To show that $\mathrm{Hol}(F_2)$ satisfies the Fibered Isomorphism Conjecture, we use the fact that every automorphism of F_2 is *geometric*, i.e. it can be realized by a diffeomorphism on a surface with boundary. That allows us to show first that $\mathrm{Aut}(F_2)$ satisfies the Conjecture and using the same fact again that $\mathrm{Hol}(F_2)$ does also.

The group $\mathrm{GL}_2(\mathbb{Z})$ splits as an amalgamated free product of finite dihedral groups:

$$\mathrm{GL}_2(\mathbb{Z}) = D_4 *_{D_2} D_6.$$

Thus both $\mathrm{Aut}(F_2)$ and $\mathrm{Hol}(F_2)$ split as amalgamated free products. Using this fact and the properties of elements of finite order in $\mathrm{Aut}(F_2)$ given in [16, 7], we determine the finite and the virtually infinite cyclic subgroups of $\mathrm{Aut}(F_2)$ and $\mathrm{Hol}(F_2)$, up to isomorphism. The list of groups is short and their lower K-theory vanishes. The Main Theorem follows from this observation.

The authors would like to thank Tom Farrell for asking the question on the lower K*-theory of* $\mathrm{Aut}(F_2)$*.*

2. Preliminaries

Let G be a discrete group. By a *class of subgroups* of G we wean a collection of subgroups of G that is closed under taking subgroups and conjugates. In our application we consider the following classes of subgroups:

- $\mathcal{F}in$, the class of finite subgroups of G.
- $\mathcal{FBC}$ for the class of finite by cyclic subgroups. Those are subgroups $H < G$ such that
$$1 \to A \to H \to C \to 1$$
where C is cyclic (finite or infinite) group and A is finite. Notice that $\mathcal{FBC}$ contains $\mathcal{F}in$ and subgroups $H = A \rtimes \mathbb{Z}$, when C is the infinite cyclic group.
- $\mathcal{VC}$ for the class of virtually cyclic subgroups of G. For a such a subgroup H either $H \in \mathcal{F}in$, $H \in \mathcal{FBC}$, or $H = A *_B C$, where A, B, C are finite and $[A : B] = [C : B] = 2$.
- $\mathcal{ALL}$ for the class of all subgroups of G.

Let $\mathcal{C}$ be a class of subgroups of G. The *classifying space* for $\mathcal{C}$, $E_{\mathcal{C}}G$ is a G-CW-complex such that the isotropy groups of the actions are in $\mathcal{C}$ and, for each $H \in \mathcal{C}$, the fixed point set of H is contractible (for more details [5], [13]).

Remark 2.1. For $\mathrm{Aut}(F_2)$ the classifying space for finite groups is the auter space ([10], [11]).

The Fibered Isomorphism Conjecture (FIC) was stated by Farrell–Jones ([8]). For the groups that holds, it provides an inductive method for computing obstruction groups in geometric topology (for a review see [14]). If G satisfies the FIC then the natural map

$$H_n^G(E_{\mathcal{VC}}G;\mathbb{KZ}^{-\infty}) \to H_n^G(E_{\mathcal{ALL}}G;\mathbb{KZ}^{-\infty}) = K_n(\mathbb{Z}G)$$

is an isomorphism for $n \leq 1$. Notice that the left hand side of the isomorphism can be computed from the virtually cyclic subgroups of G.

In general, there are "forgetful maps"

$$H_n^G(E_{\mathcal{F}in}G;\mathbb{KZ}^{-\infty}) \to H_n^G(E_{\mathcal{FBC}}G;\mathbb{KZ}^{-\infty}) \to H_n^G(E_{\mathcal{VC}}G;\mathbb{KZ}^{-\infty})$$

The difference between the class $\mathcal{F}in$ and the class $\mathcal{VC}$ is that the second class can be captured by the Waldhausen and Bass–Farrell Nil-groups of the infinite virtually cyclic subgroups. In [4] and [6], it was shown that the second map is an isomorphism. Essentially, the authors proved that the Waldhausen's Nil-groups that appear in the K-theory of virtually infinite cyclic subgroups can be detected by the Bass–Farrell Nil-groups that appear in the $\mathcal{FBC}$ class.

The FIC is known to hold for certain classes of groups. One class of interest for this paper is the class of *strongly poly-free groups*. A group Γ is called strongly poly-free if there is a filtration:

$$\Gamma = \Gamma_0 \geq \Gamma_1 \geq \cdots \geq \Gamma_n = \{1\}.$$

such that:

(1) Γ_i is normal in Γ for each i.
(2) Γ_i/Γ_{i+1} is finitely generated free for all $0 \leq i \leq n-1$.
(3) For each $\gamma \in \Gamma$ there is a compact surface S and a diffeomorphism $f : S \to S$ such that the induced homomorphism f_* on $\pi_1(S)$ is equal to c_γ in $\mathrm{Out}(\pi_1(S))$, where c_γ is the action of γ on Γ_i/Γ_{i+1} by conjugation and $\pi_1(S)$ is identified with Γ_i/Γ_{i+1} via a suitable isomorphism.

In [2] and [9] it was shown that a finite extension of a strongly poly-free group satisfies the FIC.

Remark 2.2. Let Γ be a group that satisfies (1) and (2) above. We assume that $\Gamma_i/\Gamma_{i+1} \cong F_2$. Then G is strongly poly-free. For this, let T^2 be the torus and $p = (1,1) \in T^2$. Then $\pi_1(T^2 \setminus \{p\}, x) = F_2$. In this case,

$$\mathrm{Out}(F_2) = \mathrm{Aut}(F_2)/\mathrm{Inn}(F_2) = \mathrm{GL}_2(\mathbb{Z})$$

where $\mathrm{Aut}(F_2)$ denotes the automorphism group of F_2. Let c_γ be an induced homomorphism as in (3) above. Then the image of c_γ to $\mathrm{Out}(F_2)$ can be represented by a diffeomorphism f of T^2 that fixes p. After an isotopy starting at the identity on T^2, we can assume that f fixes a small open disk D around p. Then f induces a diffeomorphism on the compact surface

$$f : T^2 \setminus D \to T^2 \setminus D,$$

that fixes the boundary. Thus $f_* = c_\gamma$ in $\mathrm{Out}(F_2)$.

Start with an exact sequence of groups.

$$1 \to A \to B \xrightarrow{r} C \to 1.$$

In the Appendix of [8], it was shown the FIC holds for B if:

- It holds for C.

- For each virtually cyclic subgroup V of C, it holds for $r^{-1}(V)$.

Using this result we show the following

Proposition 2.3. *Let*

$$1 \to F_2 \to G \xrightarrow{r} H \to 1$$

be an exact sequence. If the FIC holds for H, then it holds for G.

Proof. Using the result in [8], it is enough to show that the FIC holds for $r^{-1}(V)$, where V is a virtually cyclic subgroup of H.

If V is finite, then $r^{-1}(V)$ is a finite extension of F_2 and $r^{-1}(V)$ is a finite extension of a free group. The result follows from Remark 2.2.

If V is infinite, then V contains an infinite cyclic normal subgroup W of finite index. Then $r^{-1}(W)$ is a normal subgroup of $r^{-1}(V)$ and fits into an exact sequence:

$$1 \to F_2 \to r^{-1}(W) \to W \to 1.$$

Then there is a filtration $r^{-1}(W) > F_2 > \{1\}$, with the first quotient being an infinite cyclic group. Obviously, every homomorphism of $\mathbb{Z}$ is realized by a diffeomorphism of $S^1 \times [0,1]$. Using Remark 2.2, we see that $r^{-1}(W)$ is strongly poly-free. Therefore, $r^{-1}(V)$ is a finite extension of a strongly poly-free group. By [9], it satisfies the FIC, completing the proof of the proposition. □

Let $\mathrm{Hol}(F_2)$ denote the holomorph of F_2, namely, the universal split extension of F_2:

$$1 \to F_2 \to \mathrm{Hol}(F_2) \xrightarrow{p} \mathrm{Aut}(F_2) \to 1.$$

Notice that there is an exact sequence

$$1 \to \mathrm{Inn}(F_2) \to \mathrm{Aut}(F_2) \xrightarrow{q} \mathrm{GL}_2(\mathbb{Z}) \to 1$$

that is induced by mapping the automorphisms of F_2 to the automorphisms of its abelianization. That induces an exact sequence:

$$1 \to F_2 \to \mathrm{Aut}(F_2) \xrightarrow{q} \mathrm{GL}_2(\mathbb{Z}) \to 1.$$

Proposition 2.4. *The FIC holds for* $\mathrm{Aut}(F_2)$ *and* $\mathrm{Hol}(F_2)$.

Proof. The group $\mathrm{GL}_2(\mathbb{Z})$ contains a subgroup of finite index that is isomorphic to F_2. In fact, the following short exact sequence is known to hold, as a result of the standard action that $\mathrm{GL}_2(\mathbb{Z})$ admits on the upper half plane:

$$1 \to F_2 \to \mathrm{GL}_2(\mathbb{Z}) \to D_{12} \to 1$$

(see for example [7]). Thus the FIC holds for $\mathrm{GL}_2(\mathbb{Z})$. Now $\mathrm{Aut}(F_2)$ fits into an exact sequence:

$$1 \to F_2 \to \mathrm{Aut}(F_2) \to \mathrm{GL}_2(\mathbb{Z}) \to 1.$$

By Proposition 2.3, the FIC holds for $\mathrm{Aut}(F_2)$. Also, $\mathrm{Hol}(F_2)$ fits into an exact sequence:

$$1 \to F_2 \to \mathrm{Hol}(F_2) \to \mathrm{Aut}(F_2) \to 1.$$

By Proposition 2.3 again, the FIC holds for $\mathrm{Hol}(F_2)$. □

3. Infinite Finite-by-Cyclic Subgroups of $\mathrm{Hol}(F_2)$

Since there is an exact sequence

$$1 \to \mathrm{Inn}(F_2) \to \mathrm{Aut}(F_2) \xrightarrow{p} \mathrm{GL}_2(\mathbb{Z}) \to 1$$

and $F_2 = \langle a, b \rangle$ is torsion free, every finite subgroup of $\mathrm{Aut}(F_2)$ maps isomorphically to a finite subgroup of $\mathrm{GL}_2(\mathbb{Z})$. On the other hand, $\mathrm{GL}_2(\mathbb{Z})$ admits a decomposition as an amalgamated free product of the form

$$\mathrm{GL}_2(\mathbb{Z}) = D_4 *_{D_2} D_6 \tag{1}$$

where D_2, D_4 and D_6 are dihedral groups of orders 4, 8 and 12 respectively. Hence, any finite subgroup of $\mathrm{GL}_2(\mathbb{Z})$ is a subgroup of a conjugate of either D_2 or D_4 or D_6 and hence, so is every finite subgroup of $\mathrm{Aut}(F_2)$.

Now a presentation for $\mathrm{Aut}(F_2)$ is given by

$$\langle p, x, y, \tau_a, \tau_b \mid x^4 = p^2 = (px)^2 = 1, (py)^2 = \tau_b, x^2 = y^3\tau_b^{-1}\tau_a,$$
$$p^{-1}\tau_a p = x^{-1}\tau_a x = y^{-1}\tau_a y = \tau_b, p^{-1}\tau_b p = \tau_a, x^{-1}\tau_b x = \tau_a^{-1}, y^{-1}\tau_b y = \tau_a^{-1}\tau_b\rangle$$

where τ_a, τ_b are the inner automorphism of F_2 corresponding to a, b respectively (see for example [15]). Moreover, a presentation for $\mathrm{GL}_2(\mathbb{Z})$ is given by

$$\mathrm{GL}_2(\mathbb{Z}) = \langle P, X, Y \mid X^4 = P^2 = (PX)^2 = (PY)^2 = 1, X^2 = Y^3\rangle$$

and G maps onto $\mathrm{GL}_2(\mathbb{Z})$ by $p \mapsto P$, $x \mapsto X$, $y \mapsto Y$, $\tau_a, \tau_b \mapsto 1$.

As shown in [16], if g is an element of finite order in $\mathrm{Aut}(F_2)$, then g is conjugate in $\mathrm{Aut}(F_2)$, to one of the following elements $p, px, px\tau_a, x^2, y^2\tau_b^{-1}$ or x with orders $2, 2, 2, 2, 3$ or 4 respectively. This fact implies that $\mathrm{Aut}(F_2)$ cannot contain finite subgroups isomorphic to D_6. Moreover, any element of $\mathrm{Aut}(F_2)$ can be written uniquely in the form $p^r u(x, y)x^{2s}w(\tau_a, \tau_b)$ where $r, s \in \{0, 1\}$, $w(\tau_a, \tau_b)$ is a reduced word in $\mathrm{Inn}(F_2)$ and $u(x, y)$ is a reduced word where x, y, y^{-1} are the only powers of x, y appearing (see [16, 15]).

Also, due to the decomposition (1) of $\mathrm{GL}_2(\mathbb{Z})$, $\mathrm{Aut}(F_2)$ is also an amalgamated free product of the form

$$\mathrm{Aut}(F_2) = B *_D C \tag{2}$$

where B, C and D fit into the following short exact sequences

$$1 \to \mathrm{Inn}(F_2) \to B \to D_4 \to 1$$
$$1 \to \mathrm{Inn}(F_2) \to C \to D_6 \to 1$$
$$1 \to \mathrm{Inn}(F_2) \to D \to D_2 \to 1.$$

Moreover, since every one of B, C and D are free-by-finite groups they admit an action on a tree with finite quotient graph and finite vertex and edge stabilizers (as a corollary of the Almost Stability Theorem of Dicks and Dunwoody [7]). In fact, they are also amalgamated free products of the form

$$\begin{aligned} B &= D_4 *_{\mathbb{Z}/2\mathbb{Z}} D_2 = \langle x, p\rangle *_{\langle px\rangle} \langle px, x^2\tau_b\rangle \\ C &= D_3 *_{\mathbb{Z}/2\mathbb{Z}} D_2 = \langle y^2\tau_b^{-1}\tau_a, p\rangle *_{\langle p\rangle} \langle p, x^2\rangle \\ D &= D_2 * (\mathbb{Z}/2\mathbb{Z}) = \langle p, x^2\rangle * \langle x^2\tau_b\rangle. \end{aligned} \tag{3}$$

Once again, the elements of finite order are $p, px, x^2\tau_b, px^3\tau_b, x^2, px^2, y^2\tau_b^{-1}\tau_a$ and x. To be in accordance with Meskin, we see that $x^2(y^2t_b^{-1}\tau_a)x^2 = y^3\tau_b^{-1}\tau_a(y^2\tau_b^{-1}\tau_a)\tau_a^{-1}\tau_b y^{-3} = y^2\tau_b^{-1}$, $\tau_a^{-1}x^{-1}(px^3\tau_b)x\tau_a = px\tau_a$, $y(x^2\tau_b)y^{-1} = x^2$ and that $x^{-1}(px^2)x = p$.

By definition, $G = \mathrm{Hol}(F_2)$ is the universal split extension of $\mathrm{Aut}(F_2)$ and thus it fits to the split exact sequence

$$1 \to F_2 \to \mathrm{Hol}(F_2) \to \mathrm{Aut}(F_2) \to 1.$$

So $\mathrm{Hol}(F_2) = F_2 \rtimes \mathrm{Aut}(F_2)$. Hence, the above presentation for $\mathrm{Aut}(F_2)$ provides us with a presentation for $\mathrm{Hol}(F_2)$. Namely,

$$\begin{gathered}\mathrm{Hol}(F_2) = \langle p, x, y, \tau_a, \tau_b, a, b \mid x^4 = p^2 = (px)^2 = 1, (py)^2 = \tau_b, x^2 = y^3\tau_b^{-1}\tau_a,\\ p^{-1}\tau_a p = x^{-1}\tau_a x = y^{-1}\tau_a y = \tau_b, p^{-1}\tau_b p = \tau_a, x^{-1}\tau_b x = \tau_a^{-1}, y^{-1}\tau_b y = \tau_a^{-1}\tau_b,\\ \tau_a^{-1}a\tau_a = a, \tau_a^{-1}b\tau_a = a^{-1}ba, \tau_b^{-1}a\tau_b = b^{-1}ab, \tau_b^{-1}b\tau_b = b, p^{-1}ap = b, p^{-1}bp = a,\\ x^{-1}ax = b, x^{-1}bx = a^{-1}, y^{-1}ay = b, y^{-1}by = a^{-1}b\rangle.\end{gathered}$$

Moreover, the decomposition (2) of $\mathrm{Aut}(F_2)$ provides an amalgamated free product decomposition for $\mathrm{Hol}(F_2)$:

$$\mathrm{Hol}(F_2) = (F_2 \rtimes B) *_{F_2 \rtimes D} (F_2 \rtimes C) \tag{4}$$

and based on (3) we have

$$\begin{aligned} F_2 \rtimes B &= (F_2 \rtimes D_4) *_{F_2 \rtimes \mathbb{Z}/2\mathbb{Z}} (F_2 \rtimes D_2)\\ F_2 \rtimes C &= (F_2 \rtimes D_3) *_{F_2 \rtimes \mathbb{Z}/2\mathbb{Z}} (F_2 \rtimes D_2) \\ F_2 \rtimes D &= (F_2 \rtimes D_2) * (F_2 \rtimes \mathbb{Z}/2\mathbb{Z})\end{aligned} \tag{5}$$

Based again on the Almost Stability Theorem, we see that every vertex group in the above graphs of groups is a free-by-finite group so, it also admits a decomposition as a graph of groups with finite vertex groups. An analysis, based on the presentations and also on the fact that the action of x, y, p on a, b is the same as that on τ_a, τ_b, would give us the following: In $F_2 \rtimes B$,

$$F_2 \rtimes D_4 = D_4 *_{\mathbb{Z}/2\mathbb{Z}} D_2 = \langle x, p\rangle *_{\langle px\rangle} \langle px, x^2 b\rangle$$

$$F_2 \rtimes D_2 = D_2 *_{\mathbb{Z}/2\mathbb{Z}} D_2 *_{\mathbb{Z}/2\mathbb{Z}} D_2 = \langle px, x^2\tau_b\rangle *_{\langle px\rangle} \langle px, x^2\tau_b b^{-1}\rangle *_{\langle x^2\tau_b b^{-1}\rangle} \langle pxa, x^2\tau_b b^{-1}\rangle$$

$$F_2 \rtimes \mathbb{Z}/2\mathbb{Z} = (\mathbb{Z}/2\mathbb{Z} *_b) * \mathbb{Z}/2\mathbb{Z} = (\langle px\rangle *_b) * \langle pxa\rangle.$$

In $F_2 \rtimes C$,

$$F_2 \rtimes D_3 = D_3 *_{\mathbb{Z}/2\mathbb{Z}} D_3 = \langle y^2\tau_b^{-1}, p\rangle *_{\langle p\rangle} \langle y^2\tau_b^{-1}a, p\rangle$$

$$F_2 \rtimes D_2 = D_2 * \mathbb{Z}/2\mathbb{Z} = \langle p, x^2\rangle * \langle x^2 b\rangle$$

$$F_2 \rtimes \mathbb{Z}/2\mathbb{Z} = (\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}) *_a = (\langle p\rangle * \langle pba^{-1}\rangle) *_a .$$

Finally, in $F_2 \rtimes D$,

$$F_2 \rtimes D_2 = D_2 * \mathbb{Z}/2\mathbb{Z} = \langle p, x^2\rangle * \langle x^2 b\rangle$$

$$F_2 \rtimes \mathbb{Z}/2\mathbb{Z} = \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z} = \langle x^2\tau_b\rangle * \langle x^2\tau_b b\rangle * \langle x^2\tau_b b^{-1} a\rangle.$$

In the above, $S *_t$ denotes the HNN-extension with base group S and stable letter t.

For example, $F_2 \rtimes \langle px, x^2\tau_b\rangle$ has a presentation of the form

$$\langle \xi_1, \xi_2, a, b \mid \xi_1^2 = \xi_2^2 = 1, [\xi_1, \xi_2] = 1, \xi_1 a\xi_1 = a^{-1}, \xi_1 b\xi_1 = b, \xi_2 a\xi_2 = b^{-1}a^{-1}b, \xi_2 b\xi_2 = b^{-1}\rangle$$

where $\xi_1 = px$ and $\xi_2 = x^2\tau_b$. By setting $\zeta_2 = b\xi_2$ and eliminating b, we get

$$\langle \xi_1, \xi_2, a, \zeta_2 \mid \xi_1^2 = \xi_2^2 = \zeta_2^2 = 1, [\xi_1, \xi_2] = 1, \xi_1 a\xi_1 = a^{-1}, \zeta_2 a\zeta_2 = a^{-1}, [\xi_1, \zeta_2] = 1\rangle.$$

Now by setting $\xi_3 = \xi_1 a$ and eliminating a we get

$$\langle \xi_1, \xi_2, \xi_3, \zeta_2 \mid \xi_1^2 = \xi_2^2 = \zeta_2^2 = \xi_3^2 = 1, [\xi_1, \xi_2] = [\zeta_2, \xi_3] = [\xi_1, \zeta_2] = 1\rangle$$

which is the desired decomposition.

So now we can prove the following result which generalizes the result in [16] on the elements of finite order in $\mathrm{Aut}(F_2)$.

Lemma 3.1. *An element of finite order in* $\mathrm{Hol}(F_2)$ *is conjugate to exactly one of* $p, px, pxa, px\tau_a, px\tau_a a, x^2, x^2b, y^2\tau_b^{-1}, y^2\tau_b^{-1}a$ *and* x *with orders* 2, 2, 2, 2, 2, 2, 2, 3, 3 *and* 4 *respectively.*

Proof. Given the above decomposition, every element of finite order is a conjugate of an element of a vertex group. So it suffices to observe the following: $x^2(px^3\tau_b b)x^2 = px\tau_a a$, $x^2(px^3\tau_b)x^2 = px\tau_a$, $x^2(px^3b)x^2 = pxa$, $x(px^2)x^{-1} = p$, $y(x^2\tau_b)y^{-1} = x^2$, $b^{-1}(x^2\tau_b b^{-1})b = x^2\tau_b b$, $x\tau_a^{-1}\tau_b y^{-1}(x^2\tau_b b^{-1}a)y\tau_b^{-1}\tau_a x^{-1} = x^2\tau_b b$ and $yb^{-1}x\tau_a^{-1}\tau_b y^{-1}(x^2\tau_b b)y\tau_b^{-1}\tau_a x^{-1}by^{-1} = x^2b$. Notice also that x^2b is no longer conjugate to x^2 since the relation $x^2 = y^3\tau_b^{-1}\tau_a$ has no equivalent for a and b due to the semidirect product structure of G. □

From the fact that $\mathrm{Hol}(F_2) = \langle a, b\rangle \rtimes \mathrm{Aut}(F_2)$ we have that every element W of $\mathrm{Hol}(F_2)$ can be written uniquely in the form

$$W = Vz(a, b)$$

where $V \in \mathrm{Aut}(F_2)$ and $z(a, b)$ is a word in the free group $\langle a, b\rangle$. So, the normal form for the elements of $\mathrm{Aut}(F_2)$ implies the existence of a normal form for the elements of $\mathrm{Hol}(F_2)$:

$$W = p^r u(x, y)x^{2s} w(\tau_a, \tau_b)z(a, b)$$

where $w(t_a, t_b)$ is a reduced word in the free group $\langle t_a, t_b\rangle$, $u(x, y)$ is a reduced word where x, y, y^{-1} are the only powers of x, y appearing and $r, s \in \{0, 1\}$. Moreover, every vertex group in the decomposition (4) has also a normal form. More specifically, every element in $F_2 \rtimes B$ can be written uniquely in the form $p^r x^n x^{2s} w(\tau_a, \tau_b)z(a, b)$ where $r, n, s \in \{0, 1\}$ and every element in $F_2 \rtimes C$ can be written uniquely in the form $p^r y^n x^{2s} w(\tau_a, \tau_b)z(a, b)$ where $r, s \in \{0, 1\}$ and $n \in \{0, 1, -1\}$ and $w(\tau_a, \tau_b)$ is a reduced word in $\langle \tau_a, \tau_b\rangle$ and $z(a, b)$ is a reduced word in $\langle a, b\rangle$.

Notice]that one can define a natural epimorphism

$$\mathrm{Hol}(F_2) \to \mathrm{GL}_2(\mathbb{Z})$$

with kernel $\langle a, b\rangle \rtimes \langle \tau_a, \tau_b\rangle$. In fact, $\mathrm{Hol}(F_2)$ fits into the following short exact sequence

$$1 \to F_2 \rtimes F_2 \to \mathrm{Hol}(F_2) \to \mathrm{GL}_2(\mathbb{Z}) \to 1$$

although such a sequence does not split.

We are searching for subgroups of $G = \mathrm{Hol}(F_2)$ which are isomorphic to $A \rtimes \mathbb{Z}$ where A is a finite subgroup of G. In our argument we shall make extensive use of the following well known result of Bass-Serre theory [17]. Let M be a group that acts on its standard tree T and $m \in M$ such that m stabilizes two distinct vertices of T. Then m stabilizes the (unique reduced) path that connects the two vertices. In particular, m is an element of every edge stabilizer of every edge that constitute the path that connects the two vertices.

In fact we shall show the following:

Proposition 3.2. *The only finite-by-cyclic subgroups of* G *are* $\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}$.

Proof. **Claim 1**. The only subgroups isomorphic to $A \rtimes \mathbb{Z}$ with A finite cyclic are isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}$.

One can easily check that $\langle px, b\rangle \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}$ and so $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}$ is a subgroup group of G.

Now the only elements of order three in $\mathrm{Hol}(F_2)$ are conjugates of $y^2\tau_b^{-1}$ or $y^2\tau_b^{-1}a$. Assume that there is a subgroup of G isomorphic to $\mathbb{Z}/3\mathbb{Z} \rtimes \mathbb{Z}$. Then, conjugating if necessary, we may assume that there is an element of infinite order in G, say g, such that $g^{-1}(y^2\tau_b^{-1}a^s)g = (y^2\tau_b^{-1}a^s)^{\pm 1}$ with $s \in \{0, 1\}$. Based on the decomposition (4) we see that the above relation implies that $y^2\tau_b^{-1}a^s$ stabilizes both vertices $F_2 \rtimes C$ and $g^{-1}(F_2 \rtimes C)$ and hence the path that connects them. So it belongs to the edge stabilizer $F_2 \rtimes D$, unless $g \in F_2 \rtimes C$. But if $g \in F_2 \rtimes D$ we have a contradiction since by decomposition (5), $F_2 \rtimes D$ cannot contain elements of order 3. Now if $g \in F_2 \rtimes C$, then based again on the decomposition (5) of $F_2 \rtimes C$, we have that g stabilizes both $F_2 \rtimes D_3$ and $g^{-1}(F_2 \rtimes D_3)$ and so it belongs to $F_2 \rtimes \mathbb{Z}/2\mathbb{Z}$, a further contradiction, unless again $g \in F_2 \rtimes D_3$. Finally, by the decomposition of $F_2 \rtimes D_3 = D_3 *_{\mathbb{Z}/2\mathbb{Z}} D_3$ we have again that g has to be an element of either of the two D_3 vertices and hence an element of finite order.

We shall show now that G cannot contain subgroups isomorphic to $\mathbb{Z}/4\mathbb{Z} \rtimes \mathbb{Z}$. Assume that G contains such a subgroup, say A. Then, A is generated by a conjugate of x, since the conjugacy class of x is the only class of elements of order 4, and by an element g of G. Using conjugation if necessary, we may assume the element of order 4 in A is x. Now let $g \in G$ such that $\langle x, g\rangle \cong \mathbb{Z}/4\mathbb{Z} \rtimes \mathbb{Z}$. Then $g^{-1}xg = x^{\pm 1}$. Let G act to the tree that corresponds to the decomposition (4). Then, due to the above relation, x stabilizes both $F_2 \rtimes B$ and $g^{-1}(F_2 \rtimes B)$ and so it stabilizes the path between the two vertices. Hence, $x \in F_2 \rtimes D$ a contradiction, unless $g \in F_2 \rtimes B$. Moreover, using the decomposition (5) we see that g can only be an element of $F_2 \rtimes D_4$ and using the fact that $F_2 \rtimes D_4 = D_4 *_{\mathbb{Z}/2\mathbb{Z}} D_2$ it can only be an element of D_4 and so is of finite order. This completes the proof of claim 1.

Claim 2. There are no subgroups of G of the form $D_2 \rtimes \mathbb{Z}$ or of the form $D_3 \rtimes \mathbb{Z}$.

Up to conjugacy, the possible D_2 in G are $\langle x^2, p\rangle$, $\langle px, x^2\rangle$, $\langle px, x^2b\rangle$, $\langle px, x^2\tau_b\rangle$, $\langle px, x^2\tau_b b^{-1}\rangle$ and $\langle pxa, x^2\tau_b b^{-1}\rangle$. Now notice that all last five, $\langle px, x^2\rangle$, $\langle px, x^2b\rangle$, $\langle px, x^2\tau_b\rangle$, $\langle px, x^2\tau_b b^{-1}\rangle$ and $\langle pxa, x^2\tau_b b^{-1}\rangle$ appear only once in the graph of groups decomposition (5) of $\mathrm{Hol}(F_2)$, as vertex groups. Moreover, in all five, none of the generators is conjugate to the other, i.e. there are no $g \in G$ such that $gpxg^{-1} = x^2$ or $gpxg^{-1} = x^2b$ or $gpxg^{-1} = x^2\tau_b b^{-1}$ or $gpxg^{-1} = x^2b$ or $gpxag^{-1} = x^2\tau_b b^{-1}$ by Lemma 3.1. Hence, a relation of the form $gD_2g^{-1} = D_2$ implies (repeating again the argument of claim 1) that g is an element of finite order. So the only possibility for a semidirect product $D_2 \rtimes \mathbb{Z}$ lies with $\langle x^2, p\rangle$.

So assume that there is an element $g \in G$ such that $\langle g, p, x^2\rangle = D_2 \rtimes \mathbb{Z}$. Then, since p and x^2 are not conjugates and px^2 is conjugate to p, the action of g is either $gpg^{-1} = p$ and $gx^2g^{-1} = x^2$, or $gpg^{-1} = px^2$ and $gx^2g^{-1} = x^2$. Let us concentrate to the relation $gpg^{-1} = p$. Given the normal form of $g = p^r u(x, y)x^{2s}w(\tau_a, \tau_b)z(a, b)$ we have that

$$p^r u(x, y)x^{2s}w(\tau_a, \tau_b)z(a, b)pz^{-1}(a, b)w^{-1}(\tau_a, \tau_b)x^{-2s}u^{-1}(x, y)p^{-r} = p.$$

The above relation implies the existence of the following relation in $\mathrm{GL}_2(\mathbb{Z})$:

$$P^r U(X, Y)X^{2s}PX^{2s}U^{-1}(X, Y)P^r = P$$

which is equivalent to

$$UPU^{-1} = P.$$

By the normal form for the elements of $\mathrm{GL}_2(\mathbb{Z})$, we have that U is of the form $U = XY^{e_1} \dots XY^{e_k}$ with $e_i \in \{\pm 1\}$. So the word UPU^{-1} becomes

$$XY^{e_1} \dots XY^{e_k} PY^{-e_k} X^{-1} \dots Y^{-e_1} X^{-1} =$$

$$XY^{e_1} \dots XY^{e_k} Y^{e_k} X \dots Y^{e_1} X \cdot P =$$

$$\begin{aligned} X^2 \cdot XY^{e_1} \dots XY^{e_{k-1}} X \cdot Y^{-1} \cdot XY^{e_{k-1}} \dots Y^{e_1} X \cdot P \quad &\text{if} \quad e_k = 1 \\ X^2 \cdot XY^{e_1} \dots XY^{e_{k-1}} X \cdot Y \cdot XY^{e_{k-1}} \dots Y^{e_1} X \cdot P \quad &\text{if} \quad e_k = -1 \\ XY^{e_1} \dots XY^{\pm 1} X \dots Y^{e_1} XP \quad &\text{if} \quad e_k = 0 \text{ and } e_{k-1} = \mp 1. \end{aligned}$$

In all cases, the relation $UPU^{-1} = P$ is impossible, since, after deletions of P, the remaining word is reduced as written so is never trivial, unless $U = 1$. Hence $u = 1$ and so the only possible $g = p^r x^{2s} w(\tau_a, \tau_b) z(a, b)$. Then the relation $gpg^{-1} = p$ gives

$$x^{2s} w(\tau_a, \tau_b) z(a, b) p z^{-1}(a, b) w^{-1}(\tau_a, \tau_b) x^{2s} = p$$

i.e. $w(\tau_b, \tau_a) w^{-1}(\tau_a, \tau_b) = 1$. One can easily see that if $w(\tau_a, \tau_b)$ is a reduced word in $\mathrm{Inn}(F_2) \cong F_2$ then the word $w(\tau_b, \tau_a)$ is reduced and the word $w(\tau_b, \tau_a) w^{-1}(\tau_a, \tau_b)$ is reduced and cyclically reduced as written. Hence, a relation $w(\tau_b, \tau_a) w^{-1}(\tau_a, \tau_b) = 1$ is impossible unless $w = 1$. That implies $z(b, a) z^{-1}(a, b) = 1$ and again $z = 1$. Then $g = p^r x^{2s}$ which has finite order for all possible r, s.

Let us now examine the possibility $gpg^{-1} = px^2$. This implies that

$$p^r u(x, y) x^{2s} w(\tau_a, \tau_b) z(a, b) p z^{-1}(a, b) w^{-1}(\tau_a, \tau_b) x^{-2s} u^{-1}(x, y) p^{-r} = px^2.$$

Projection to $\mathrm{GL}_2(\mathbb{Z})$ gives

$$P^r U(X, Y) X^{2s} P X^{2s} U^{-1}(X, Y) P^r = PX^2$$

which is equivalent to

$$UPU^{-1} = PX^2.$$

Performing the same analysis as above for UPU^{-1} we get that the only possibility is $U = X$, hence $u = x$. Then $g = p^r x x^{2s} w(\tau_a, \tau_b) z(a, b)$ and so the relation $p^r x x^{2s} w(\tau_a, \tau_b) z(a, b) p z^{-1}(a, b) w^{-1}(\tau_a, \tau_b) x^{2s} x^{-1} p^r = px^2$ implies again that $w(\tau_b, \tau_a) w^{-1}(\tau_a, \tau_b) = 1$ which possible if and only if $w = 1$ and so $z(b, a) z^{-1}(a, b) = 1$ which is possible if and only if $z = 1$.

Finally, one can easily check that existence of $g \in G$ such that $gD_3 g^{-1} = D_3$ can only occur for g of finite order (using again the previous arboreal argument), so we have that no subgroup of the form $D_3 \rtimes \mathbb{Z}$.

The only case left is subgroups isomorphic to $D_4 \rtimes \mathbb{Z}$. It is easy to see that then it will contain subgroups isomorphic to $D_2 \rtimes \mathbb{Z}$, which is impossible. □

4. Vanishing of lower K-theory of $\mathrm{Hol}(F_2)$

We will prove the main result of the paper. For a group G, we write

$$\mathrm{Wh}_q(G) = \begin{cases} \mathrm{Wh}(G), & \text{if } q = 1, \\ \tilde{K}_0(\mathbb{Z}G), & \text{if } q = 0, \\ K_q(\mathbb{Z}G), & \text{if } q < 0. \end{cases}$$

Theorem 4.1. *Let $\Gamma < \mathrm{Hol}(F_2)$. Then for all $q \leq 1$, $Wh_q(\Gamma) = 0$.*

Proof. We will show the theorem for $G = \mathrm{Hol}(F_2)$. The proof for $\mathrm{Aut}(F_2)$ is similar. By Proposition 2.4, G satisfies the FIC. Let $\Gamma < \mathrm{Hol}(F_2)$. Then by [8], Γ also satisfies the FIC. Thus the maps

$$H_q^G(E_{\mathcal{FBC}}\Gamma; \mathbb{K}\mathbb{Z}^{-\infty}) \to \mathrm{Wh}_q(\mathbb{Z}\Gamma), \qquad q \leq 1,$$

are isomorphisms. There is a spectral sequence that computes the left hand side of such an isomorphism:

$$E_{i,j}^2 = H_i^G(E_{\mathcal{FBC}}\Gamma; \mathrm{Wh}_j(V)) \Longrightarrow \mathrm{Wh}_{i+j}(\Gamma),$$

where V is in $\mathcal{FBC}$. Now, by the decomposition of $\mathrm{Hol}(F_2)$ and Proposition 3.2:

(1) If V is finite, V will isomorphic to one of the following groups: $\mathbb{Z}/2\mathbb{Z}$, $\mathbb{Z}/3\mathbb{Z}$, $\mathbb{Z}/4\mathbb{Z}$, D_2, D_4. But in this case from the lists in [1] and [12]

$$\mathrm{Wh}_q(V) = 0, \qquad \text{for } q \leq 1.$$

(2) If V is infinite, then $V = \mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}$. Using the Bass–Heller–Swan Formula and the calculations of the Nil-groups in [3], we have that:

$$\mathrm{Wh}_q(V) = 0, \qquad \text{for } q \leq 1.$$

Thus $\mathrm{Wh}_q(\Gamma) = 0$ for all $q \leq 1$. □

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